NONSTATIONARY SHEAR FLOW OF A VISCOPLASTIC MEDIUM

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We consider problems involving nonstationary shear flow of a viscoplastic medium between two parallel plates and also in a cylindrical tube under the action of a time-varying shear stress applied to the walls of the passage.

A survey of studies relating to the hydrodynamics of viscoplastic media (Schwedoff–Bingham plastics) is contained in [1-5]. In the analysis of shear flows of viscoplastic media the greatest interest centers on finding the surface separating the viscous flow zone from the zone of quasirigid motion; this leads to problems with an unknown boundary for an equation of parabolic type, analogous to freezing problems [6, 7].

In [8] nonstationary flows of a viscoplastic medium were investigated by using a method of statistical experiments (Monte Carlo method).

Problems concerning freezing were studied in [9-13] by the method of successive approximations; a series of such problems were solved in this way.

We consider a nonstationary one-dimensional shear flow of a viscoplastic medium in a two-dimensional channel of height $2a$ or in a cylindrical tube of cross-section radius a under the action of a time-varying shear stress applied to the walls of the two-dimensional passage or to the wall of the cylindrical tube.

The flow picture assumed here, together with the arrangement of the coordinate system, is shown in Fig. 1.

We assume that for $t < 0$ the walls of the passage ($x = \pm a$) or the wall of the cylindrical tube ($x = a$) are fixed; the motion of the medium commences at the instant $t = 0$ from a state of rest; for $t > 0$ the flow has a unique nonzero velocity component $u_z = u(x, t)$, and the tangential shear stress τ is a function only of the transverse coordinate x and the time t. The rheological equation of the flow of a viscoplastic medium (Schwedoff–Bingham plastics) for one-dimensional flows of this type has the form

$$\tau = \mu \frac{\partial u}{\partial x} + \tau_0 \operatorname{sign}\left(\frac{\partial u}{\partial x}\right) \quad \text{for} \quad |\tau| \geqslant \tau_0 \tag{1}$$

$$\frac{\partial u}{\partial x} = 0 \quad \text{for} \quad |\tau| < \tau_0 \tag{2}$$

where μ is the coefficient of dynamic viscosity, τ_0 is the limiting tangential shear stress (rheological constants of the medium). By taking advantage of the symmetry of the two-dimensional shear flow we can restrict our discussion to the upper half of the passage, $0 < x \le a$. Moreover, in both the two-dimensional and axially symmetric cases, we assume that the condition

$$\operatorname{sign}\left(\frac{\partial u}{\partial x}\right) = 1$$

is satisfied in the viscous flow zone for all values of $t > 0$.

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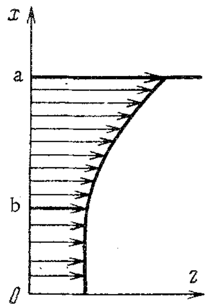

Fig. 1

In the absence of a pressure gradient and a volume density of external forces we can, upon taking into account the assumptions made above, write the equation of motion of the continuous medium in the form

$$\rho \frac{\partial u}{\partial t} = \frac{1}{x^k} \frac{\partial}{\partial x} (x^k \tau) \tag{3}$$

where ρ is the density of the medium, and k = 1 for the two-dimensional case while k = 2 for the axially symmetric case.

Differentiating Eq. (3) with respect to the variable x and Eq. (1) with respect to variable t, and eliminating the expression $\partial^2 u/\partial x \partial t$ from the resulting system, we obtain

$$\frac{\rho}{\mu} \frac{\partial \tau}{\partial t} = \frac{\partial}{\partial x} \left[\frac{1}{x^k} \frac{\partial}{\partial x} (x^k \tau) \right] \tag{4}$$

We remark that Eq. (4) describes the time variation of the tangential shear stress distribution in the viscous flow zone since it is only in this zone that the expressions (1) and (3) are valid.

By virtue of the continuity of the tangential shear stresses at x = a, we have the boundary condition

$$\tau (a, t) = \varphi (t) \tag{5}$$

If x = δ(t) is the equation of the boundary separating the plastic flow zone from the zone of quasirigid motion, then when x = δ(t) the following condition must be satisfied on this unknown boundary:

$$\tau (x, t) = \tau_0 \quad \text{for} \quad x = \delta (t) \tag{6}$$

As a consequence of the motion of the quasirigid core as a single entity, we have

$$\frac{1}{x^k} \frac{\partial}{\partial x} (x^k \tau) = (k + 1) \frac{\tau_0}{\delta (t)} \quad \text{for} \quad x = \delta (t) \tag{7}$$

Since the flow develops from a state of rest in which the quasirigid zone occupies the whole flow region, we have, as our initial condition on δ(t),

$$\delta (0) = a \tag{8}$$

We now develop relation (7) in more detail, since neglecting this condition would lead to erroneous results [14, 15], as shown for the two-dimensional case in [16]. We consider the motion of the viscoplastic medium in the quasirigid zone, for which, taking into account the rheological equation in the form (2), we have u = u_0(t). Moreover from the equation of motion (3) it follows that

$$\frac{1}{x^k} \frac{\partial}{\partial x} (x^k \tau) = f (t), \ 0 < x < \delta (t) \tag{9}$$

where f(t) is a function yet to be defined.

The general solution of Eq. (9) has the form

$$\tau = f (t) \, x \, (k + 1)^{-1} + x^{-k} \, C (t) \tag{10}$$

where C(t) is an arbitrary function. When x → 0 the tangential shear stress $\tau \to 0$; therefore C(t) = 0. Taking into account the condition (6) on the unknown boundary, we obtain from expression (10)

$$\tau = \tau_0 \, x / \delta (t) \tag{11}$$

The relation (7) can be obtained from the expression (11) by taking into account the continuity of the velocity of the medium and the tangential shear stresses in passing through the boundary x = δ(t) separating the zones. We introduce the dimensionless quantities

$$x_* = \frac{x}{a}, \quad \Delta_* = \frac{\delta}{a}, \quad \tau_* = \frac{\tau}{T}, \quad S = \frac{\tau_0}{T}, \quad t_* = \frac{\mu}{\rho a^2} t$$
$$\varphi_* (t_*) = \frac{a \varphi (t)}{T}$$

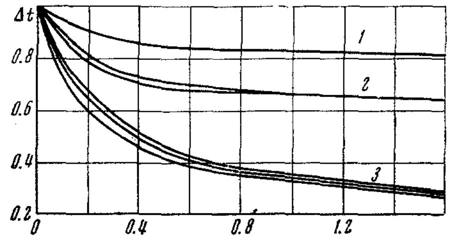

Fig. 2

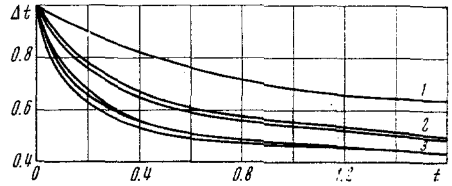

Fig. 3

where T is a characteristic stress. In dimensionless form problem (4) and (8) has the form

$$\frac{\partial\tau}{\partial t}=\frac{\partial}{\partial x}\left[\frac{1}{x^k}\frac{\partial}{\partial x}(x^k\tau)\right],\quad \Delta(t)<x<1,\quad 0<t<t_0<\infty \tag{12}$$

$$\tau(1,t)=\varphi(t) \tag{13}$$

$$\tau(\Delta,t)=S \tag{14}$$

$$\frac{1}{x^k}\frac{\partial}{\partial x}(x^k\tau)\,|_{x=\Delta}=\frac{(k+1)S}{\Delta(t)} \tag{15}$$

$$\Delta(0)=1 \tag{16}$$

Here and henceforth the asterisk, used to denote dimensionless quantities, will be dropped.

We now construct a solution of the problem (12)–(16). Integrating Eq. (12) twice with respect to the variable x between the limits of Δ and x and noting the conditions (15) and (14), and subsequently the condition (13) also, we obtain, after some simple manipulations, a system of functional equations for determining $\tau(x,t)$ and $\Delta(t)$:

$$\tau=S\frac{x}{\Delta}+\frac{1}{x^k}\int_\Delta^x x^k\int_\Delta^x\frac{\partial\tau}{\partial t}\,dxdx \tag{17}$$

$$\Delta=S\left[\varphi(t)-\int_\Delta^1 x^k\int_\Delta^x\frac{\partial\tau}{\partial t}\,dxdx\right]^{-1} \tag{18}$$

The functional equation (18) is compatible with the initial condition (16) on $\Delta(t)$; it places the following restriction on the function $\varphi(t)$:

$$\varphi(0)=S$$

We note that the formation of the viscoplastic flow zone is only possible when the following condition is satisfied:

$$\varphi(t)>S-\int_\Delta^1 x^k\int_\Delta^x\frac{\partial\tau}{\partial t}\,dxdx$$

For a specified class of functions $\varphi(t)$ the system of functional equations (17) and (18) may be solved by the method of successive approximations

$$\begin{gathered}\tau_{n+1}=S\frac{x}{\Delta_n}+\frac{1}{x^k}\int_{\Delta_n}^x x^k\int_{\Delta_n}^x\frac{\partial\tau_n}{\partial t}\,dxdx\\ \Delta_{n+1}=S\left[\varphi(t)-\int_{\Delta_n}^1 x^k\int_{\Delta_n}^x\frac{\partial\tau_n}{\partial t}\,dxdx\right]^{-1}\end{gathered} \tag{19}$$

As the zeroth approximation we take

$$\tau_0 = S \frac{x}{\Delta_0}, \qquad \Delta_0 = \frac{S}{\varphi(t)} \tag{20}$$

The choice (20) for the zeroth approximation corresponds physically to the case of a viscoplastic medium with an infinitely small value of the density.

Using the iterational scheme (19) with k = 0 and 1, we carried out the zeroth, first, and second approximations. To obtain concrete results we used the following expression for the function $\varphi(t)$:

$$\varphi(t) = \frac{S(1+mt)}{1+mst}$$

The function $\varphi(t)$ satisfies the conditions

$$\varphi(0) = S, \qquad \varphi(\infty) = 1$$

which makes it possible to model the outflow rate of the nonstationary flow of the viscoplastic medium on a special "stationary regime" wherein the accelerations of the channel wall and of quasirigid core of the flow stay constant with time. The presence in the expression for $\varphi(t)$ of the parameter m makes it possible to estimate the influence of the rapidity of growth of the shear stresses applied to the wall of the two-dimensional passage and to the wall of the cylindrical tube on the variation in the position of the boundary separating the zones. In addition, the parameter m has an influence on the convergence of the iterational process. Our calculations show that the convergence of the iterational process is entirely satisfactory for $1 \le m \le 10$ and that it becomes poorer as m increases. For values of $m > 100$ the process no longer converges. As the plasticity parameter S varies from 0.2 to 0.8, the convergence of the iterational process improves as S increases.

In Fig. 2 we present the results of our calculations, using the iterational scheme (19), for the two-dimensional case with m = 5 (the labels 1, 2, and 3 correspond to values of the plasticity parameter S equal, respectively, to 0.8, 0.6, and 0.2). In the curves of Fig. 2

$$\Delta_0 < \Delta_2 < \Delta_1$$

Figure 3 shows the influence of the parameter m on the development of the flow of the viscoplastic medium in a circular tube. The plasticity parameter S = 0.4 (the labels 1, 2, and 3 correspond, respectively, to values of the parameter m equal to 1, 3, and 7). In the curves of Fig. 3

$$\Delta_0 < \Delta_2 < \Delta_1$$

The maximum difference between the second and the first approximations for $\Delta(t)$ in the time interval of variation studied amounted to no more than 0.07, which confirms the applicability of our method to the solution of the problem considered.

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